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Systematics of quarter BPS operators in $\mathcal{N} = 4$ SYM*

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Abstract: A systematic construction is presented of 1/4 BPS operators in $\mathcal{N} = 4$ superconformal Yang-Mills theory, using either analytic superspace methods or components. In the construction, the operators of the classical theory annihilated by 4 out of 16 supercharges are arranged into two types. The first type consists of those operators that contain 1/4 BPS operators in the full quantum theory. The second type consists of descendants of operators in long unprotected multiplets which develop anomalous dimensions in the quantum theory. The 1/4 BPS operators of the quantum theory are defined to be orthogonal to all the descendant operators with the same classical quantum numbers. It is shown, to order g^2, that these 1/4 BPS operators have protected dimensions.

Keywords: AdS-CFT and dS-CFT Correspondence, Superspaces.

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Contents

1. Introduction

There are few things one can calculate exactly in the four-dimensional superconformal Yang-Mills theories. One class of such quantities consists of a set of correlation functions of the Bogomolnyi Prasad Sommerfield (BPS) operators. In $\mathcal{N} = 4$ superconformal Yang-Mills, there are 1/2 BPS, 1/4 BPS and 1/8 BPS operators, which are operators that are invariant under 8, 4 and 2 (out of 16) Poincaré supercharges respectively. Based on very general arguments involving only the supersymmetry algebra [1, 2, 3], the anomalous dimension of any of these operators vanishes identically in the full quantum theory.

In the AdS/CFT correspondence, the local gauge invariant operators of $\mathcal{N} = 4$ superconformal Yang-Mills are mapped to the physical states of the type-IIB superstring on $AdS_5 \times S^5$. (See [4, 5, 6] for the original papers and [7, 8, 9] for reviews.) The single trace 1/2 BPS operators (also referred to as chiral primary operators or CPOs) play a special role as they are in one-to-one correspondence with the short multiplets of supergravity and Kaluza-Klein states with spins ≤ 2. Driven by the success of this correspondence, several authors have derived non-renormalization results for various correlation functions of these operators. Results on the perturbative non-renormalisation of 2- and 3-point functions were derived in [10, 11, 12] in components and in superspace in [13, 14]; for further references on 3-point computations see [15]. An argument for the complete non-renormalisation of the 3-point function of the supercurrent multiplet based on anomalies was given in [11] and a superspace version of this was presented in [14]. An argument for the (non-perturbative) non-renormalisation of all 2- and 3-point functions of BPS operators based on an extra $U(1)_Y$ symmetry was given in [16] and this was verified using analytic superspace methods in [17] following on from the earlier work of [18]. Generalizations to n-point functions were obtained for extremal [19, 20, 22] and near-extremal correlators [23]–[27].

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Other BPS operators are also important, both from the perspective of superconformal Yang-Mills theory, and from that of the AdS/CFT correspondence. The simplest generalization is to multi-trace 1/2 BPS operators [28], for which non-renormalization results are the same as for single trace 1/2 BPS operators; see also [29]. Indeed, the arguments of [16] and [17] apply in this case too.

A more delicate generalization is to the multi-trace scalar operators obeying a 1/4 BPS shortening rule. A general group theoretic classification of such operators in free field theory was amongst the results derived in [30]. A more detailed study carried out in [31] revealed that in the full quantum interacting theory, the true 1/4 BPS operators involve admixtures of classical 1/4 BPS operators of [30] with descendants of non-BPS operators that occur in long supersymmetry multiplets. Using the operators obtained in [31], 2- and 3-point functions involving 1/2 and 1/4 BPS operators were computed in [32, 33]. In all cases studied, these correlators were shown to be non-renormalized to order g^2 as well as to be in accord with their large-N, large g^2N limit accessible through the AdS/CFT correspondence. In the framework of analytic superspace it turns out that the 1/4 BPS operators are described as tensor superfields carrying superindices. In [34] the 2- and 3-point functions of 1/4 BPS operators were analysed in analytic superspace and shown to be non-renormalised in a similar fashion to correlators of 1/2 BPS operators.[1] Correlation functions with $n \geq 4$ operators are, in general, expected to receive quantum corrections, just as the multipoint functions of 1/2 BPS operators do [35]. (See also [7, 9] for further references.)

However, an important puzzle arose in [31]. It was argued that the number of 1/4 BPS operators in the interacting quantum theory for some representations was smaller than one would have expected from counting the number of 1/4 BPS operators in the classical theory. This would have required the presence of some form of superconformal anomaly, which is not expected to be present. Also, the procedure for constructing the candidate operators used in [31] was somewhat ad hoc and difficult to generalize.

In this note we use the machinery of (4,1,1) harmonic superspace to describe 1/4 BPS operators. The use of extended supersymmetry dramatically simplifies the counting and construction of scalar composite operators in the $[q,p,q]$ representations of the R-symmetry group SU(4). We find that some operators were overlooked in [31]. Taking these operators into account eliminates the mismatch between the number of 1/4 BPS operators in the free and the interacting theory.

The construction of all 1/4 BPS operators in the fully interacting quantum theory is carried out as follows. First, in the classical interacting theory, a basis is produced of all the scalar operators in the representations of the R-symmetry group SU(4) suitable for 1/4 BPS operators with Dynkin labels $[q,p,q]$, $q \geq 1$ and with classical dimension $p+2q$. In the classical interacting theory there are natural candidates for 1/4 BPS operators as well as other operators which can be identified as descendants. The basis may be regrouped into operators of these two types. Harmonic superspace techniques reduce this

[1] In fact, this result can be extended to many protected operators, including those which are in shortened series A representations [34].

construction down to elementary group theory. The (4,1,1) superspace notation also makes distinguishing candidate 1/4 BPS operators from descendant operators simple.

Already at the Born level, the two types of operators usually mix; the overlaps between descendants and candidate BPS operators are generally nonzero. In a given representation $[q,p,q]$ of SU(4), we identify the 1/4 BPS operators as those linear combinations of candidate 1/4 BPS operators and descendants which at Born level are orthogonal to all descendant operators in the same representation of SU(4). By construction therefore, the admixture coefficients depend on N but not on the coupling g^2. A scalar composite operator in the $[q,p,q]$ of SU(4) which is annihilated by 4 out of 16 (Poincaré) supercharges can be either $(Q^2\bar{Q}^2)$-descendants of long operators or 1/4-BPS primaries.[2] Thus, we argue that after subtracting off all the descendant pieces we should be left with a protected operator. We then proceed to calculate the two-point functions involving the 1/4 BPS operators thus constructed. Remarkably, even though the renormalization of long operators is governed by interactions, our 1/4 BPS operators constructed using Born level two-point functions remain orthogonal to all the (descendants of) long operators at order g^2. We find that any two point function involving a 1/4 BPS operator constructed this way receives no order g^2 corrections.

In the following we first review the 1/4 BPS operators from the point of view of harmonic superspace and then discuss the diagonalization of the operators corresponding to the same representations that were discussed in [31].

2. $\mathcal{N}=4$ SYM in harmonic superspace

The leading component fields of the 1/4 BPS multiplets in $\mathcal{N}=4$ SCFT are given by scalar fields which transform under the internal symmetry group SU(4) in representations with Dynkin labels of the form $[q,p,q]$. The complete supermultiplets can be very simply described in harmonic superspace, and we briefly recall how this construction works.

For $\mathcal{N}$ extended supersymmetry in four dimensions, $(\mathcal{N},\mathrm{p},\mathrm{q})$ harmonic superspace is obtained from ordinary Minkowski superspace M by the adjunction of a compact manifold of the form $K \equiv H\backslash\,\mathrm{SU}(\mathcal{N})$ where $H = S(\mathrm{U(p)}\times\mathrm{U}(\mathcal{N}-(\mathrm{p}+\mathrm{q}))\times\mathrm{U(q)})$. This construction allows one to construct p projections of the $\mathcal{N}$ supercovariant derivatives $D_{\alpha i}$ and q projections of their conjugates $\bar{D}^i_{\alpha'}$ which mutually anticommute. We can therefore define generalized chiral or G-analytic superfields (G for Grassmann) in such superspaces which are annihilated by these derivatives. Now the superfields in harmonic superspace will also depend on the coordinates of K, and as this space is a complex manifold, they can be analytic in the usual sense (H-analytic) in their dependence on these coordinates. As the internal manifold is compact H-analytic fields will have finite harmonic expansions. The Lorentz scalar superfields which are both G-analytic and H-analytic, which we shall refer to as analytic, are the fields we are interested in. They can be shown to carry short irreducible unitary representations of the superconformal group (provided that they transform under irreducible representations of H). For original papers and detailed accounts of harmonic and analytic superspaces, see for example [36]–[39].

[2] It was shown in [31] that these are the only possibilities by group theory.

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2.1 (4,1,1) harmonic superspace

For the 1/4 BPS operators in $\mathcal{N} = 4$ the most appropriate harmonic superspace has $(\mathcal{N}, \mathrm{p}, \mathrm{q}) = (4, 1, 1)$. Since the analytic fields in this space will be annihilated by one D and one $\bar{D}$ it follows that they will only depend at most on 3/4 of the odd coordinates of M. Instead of working directly on the coset defined by the isotropy group $H = S(\mathrm{U}(1) \times \mathrm{U}(2) \times \mathrm{U}(1))$ we shall follow the standard practice of working on the group SU(4) which amounts to the same thing provided that all the fields have their dependence on H fixed. We denote an element of SU(4) by $u_I{}^i$ and its inverse by $(u^{-1})_i{}^I$. The group H is taken to act on the capital index I which we decompose as $I = (1, r, 4),\ r \in \{2, 3\}$, while SU(4) acts on the small indices $i, j, \ldots$. Using u and its inverse we can convert SU(4) indices into H indices and vice versa. Thus we can define

$$\begin{aligned} D_{\alpha I} &\equiv u_I{}^i D_{\alpha i} = (D_{\alpha 1}, D_{\alpha r}, D_{\alpha 4}) \\ \bar{D}^I_{\alpha'} &\equiv \bar{D}^i_{\alpha'} \left(u^{-1}\right)_i{}^I = \left(\bar{D}^1_{\alpha'}, \bar{D}^r_{\alpha'}, \bar{D}^4_{\alpha'}\right). \end{aligned} \tag{2.1}$$

Clearly, we have $\{D_{\alpha 1}, \bar{D}^4_{\beta'}\} = 0$. To differentiate in the coset space directions we use the right-invariant vector fields on SU(4) which we denote by $D_I{}^J$, and which satisfy $\bar{D}^I{}_J = -D_J{}^I$ and $D_I{}^I = 0$. These derivatives obey the Lie algebra relations of $\mathfrak{su}(4)$ and act on $u_K{}^k$ by

$$D_I{}^J u_K{}^k = \delta_K{}^J u_I{}^k - \frac{1}{4} \delta_I{}^J u_K{}^k\,. \tag{2.2}$$

The basic differential operators on SU(4) can be divided into three sets: the derivatives $(D_1{}^1, D_r{}^s, D_4{}^4)$ correspond to the isotropy group, the derivatives $(D_1{}^r, D_1{}^4, D_r{}^4)$ can be thought of as essentially the components of the $\bar{\partial}$ operator on K and the derivatives $(D_r{}^1, D_4{}^1, D_4{}^r)$ are the complex conjugates of these. Note that the derivatives $(D_1{}^r, D_1{}^4, D_r{}^4)$ commute with $D_{\alpha 1}$ and $\bar{D}^4_{\alpha'}$. G-analytic fields are annihilated by $D_{\alpha 1}$ and $\bar{D}^4_{\alpha'}$, H-analytic fields are annihilated by $(D_1{}^r, D_1{}^4, D_r{}^4)$ and analytic fields are annihilated by both of these sets of operators.

The $\mathcal{N} = 4$ Yang-Mills theory is described in Minkowski superspace by a scalar superfield $W_{ij} = -W_{ji}$ which transforms under the six-dimensional representation of SU(4) and also under the adjoint representation of the gauge group which we take to be SU(N). It is real in the sense that $\bar{W}^{ij} = \frac{1}{2}\epsilon^{ijkl} W_{kl}$. This superfield satisfies the constraints

$$\begin{aligned} \nabla_{\alpha i} W_{jk} &= \epsilon_{ijkl} \Lambda_\alpha{}^l \\ \bar{\nabla}^i_{\alpha'} W_{jk} &= 2\delta^i_{[j} \bar{\Lambda}_{\alpha' k]} \end{aligned} \tag{2.3}$$

where Λ is a superfield whose leading component is the spinor field of the multiplet.and where $\nabla_{\alpha i}$ is a spinorial derivative which is covariant with respect to the gauge group. Using the superspace Bianchi identities one can easily show that the only other independent spacetime component of W is the spacetime Yang-Mills field strength and that all of the component fields satisfy their equations of motion.

In $(4, 1, 1)$ superspace we can define the superfield $W_{1r} \equiv u_1{}^i u_r{}^j W_{ij}$. Using the properties outlined above one can easily show that

$$\nabla_{\alpha 1} W_{1r} = \bar{\nabla}^4_{\alpha'} W_{1r} = 0 \tag{2.4}$$

and that the derivatives $(D_1{}^r, D_1{}^4, D_r{}^4)$ all annihilate W_{1r}, so that W_{1r} is a covariantly analytic field on $(4,1,1)$ harmonic superspace. However, if we consider gauge-invariant products of W's, i.e. traces or multi-traces, the resulting objects will be analytic superfields; they will be annihilated by $D_{\alpha 1}$ and $\bar{D}^4_{\alpha'}$ rather than the gauge-covariant versions. These are the superfields which we shall use to construct the 1/4 BPS states. To make the formulae less cluttered we shall abbreviate[3] W_{1r} to W_{r-1} and define $W^r \equiv \epsilon^{rs} W_s$.

2.2 Quarter BPS operators

These superfields are easy to describe. The superfield corresponding to the representation $[q,p,q]$ contains $p+2q$ powers of W in the representation p of SU(2) (this is the SU(2) in the isotropy group), i.e. it has p symmetrized SU(2) indices. If $q=0$ the single trace operators are the chiral primaries which are 1/2 BPS. These operators we will refer to as CPOs and denote by A_p,

$$A_{r_1 \ldots r_p} \equiv \mathrm{tr}\left(W_{(r_1} \ldots W_{r_p)}\right) \tag{2.5}$$

The lowest CPO is the stress-tensor multiplet $T_{rs} = A_{rs}$. We can obtain further 1/2 BPS operators by taking products of CPOs and symmetrizing on all of the SU(2) indices. The 1/4 BPS operators (for $q > 0$) fall into two classes. There are operators that can be constructed as products of the CPOs with at least one pair of contracted indices, for example

$$T_{rs} T^{rs}\,, \qquad A_{rst} A^{st}\,, \qquad A_{rst} A^t{}_{uvw}\,, \qquad \text{and so on}\,. \tag{2.6}$$

These operators have no commutators in their definition, and so are the candidate 1/4 BPS operators, up to subtleties which we shall come to in due course. Operators in the other class have at least one single-trace factor in which the indices of two or more pairs of W's are contracted, as in

$$\mathrm{tr}\, W^2 W^2\,, \qquad A_{rst}\, \mathrm{tr}\, W^2 W^2\,, \qquad \mathrm{tr}\, W_r W_s W^2 W^2\,, \qquad \text{etc.} \tag{2.7}$$

where

$$W^2 \equiv W_r W^r = \epsilon^{rs} W_r W_s = \frac{1}{2}\epsilon^{rs}[W_r, W_s]\,. \tag{2.8}$$

These operators are descendants; the superspace Bianchi identities imply that

$$\epsilon^{\alpha\beta} \nabla_{\alpha i} \nabla_{\beta j} \bar{W}^{kl} = 2\delta_j{}^{[k} \left[W_{im}, \bar{W}^{l]m}\right]. \tag{2.9}$$

From this formula and its conjugate one can see that W^2 can be written as

$$W^2 = (\nabla_1)^2 \bar{W}^{14} = -\left(\bar{\nabla}^4\right)^2 W_{14} \tag{2.10}$$

where $(\nabla_1)^2 \equiv 1/2 \epsilon^{\alpha\beta} \nabla_{\alpha 1} \nabla_{\beta 1}$, $(\bar{\nabla}^4)^2 \equiv -1/2 \epsilon^{\alpha'\beta'} \bar{\nabla}^4_{\alpha'} \bar{\nabla}^4_{\beta'}$ and where $W_{14} = u_1{}^i u_4{}^j W_{ij}$. Given a product of W_r's containing a factor of W^2, therefore, the latter can be written in terms of derivatives as above and the derivatives can be taken to act on the whole expression with W^2 replaced by either W_{14} or its conjugate. This follows by G-analyticity. Indeed, if

[3] So the index r takes the values $r = 2,3$ for W_{1r}, and $r = 1,2$ for W_r.

there are two factors of W^2 in an operator then all four derivatives can be brought outside. This is because

$$\nabla_{\alpha 1} W_{14} = 0 \qquad \text{and} \qquad \bar\nabla^4_{\alpha'} \bar W^{14} = 0\,. \tag{2.11}$$

In fact, the descendant 1/4 BPS operators always have at least two factors of W^2 so that they can be written explicitly as derivatives of long operators by these means. However, we are not quite finished yet because the ancestor operators will not be H-analytic on $(4,1,1)$ harmonic superspace as they stand. This can be remedied by noting that

$$(\nabla_1)^2 \left(\bar\nabla^4\right)^2 \bar W^{1r} = \left[W^2, W^r\right] \tag{2.12}$$

with the aid of which we can write, for example, $\mathrm{tr}(W^2W^2)$ as

$$\begin{aligned}
\mathrm{tr}\left(W^2W^2\right) &= -\frac{1}{3}(\nabla_1)^2 \left(\bar\nabla^4\right)^2 \left(\mathrm{tr}\left(W_{14}\bar W^{14}\right) + \mathrm{tr}\left(W_{1r}\bar W^{1r}\right)\right) \\
&= -\frac{1}{12}(\nabla_1)^2 \left(\bar\nabla^4\right)^2 \mathrm{tr}\left(W_{ij}\bar W^{ij}\right) \\
&= -\frac{1}{12}(D_1)^2 \left(\bar D^4\right)^2 \mathrm{tr}\left(W_{ij}\bar W^{ij}\right)
\end{aligned} \tag{2.13}$$

where $D_{\alpha 1}D_{\beta 1} = \epsilon_{\alpha\beta}(D_1)^2$, and $\bar D^4_{\alpha'}\bar D^4_{\beta'} = -\epsilon_{\alpha'\beta'}(\bar D^4)^2$. Hence we see that $\mathrm{tr}(W^2W^2)$ is a descendant of the Konishi operator $K \equiv \mathrm{tr}(W_{ij}\bar W^{ij})$. In general, one can use (2.4) , (2.10), (2.11) and (2.12) to find

$$\frac{1}{2}(\nabla_1)^2 \left(\bar\nabla^4\right)^2 W_{ij} A \bar W^{ij} = W_r A \left[W^2, W^r\right] + \left[W^2, W^r\right] A W_r - 2W^2 A W^2 \tag{2.14}$$

provided A involves only the W_r. And since we are dealing with gauge invariant operators, we can replace the ∇ by D. We note for future use that each of the descendants that we consider below can be written as an ancestor superfield acted on by the differential operator $(D_1)^2(\bar D^4)^2$.

On the other hand, the CPOs themselves cannot be obtained by differentiation from other operators and so the candidate 1/4 BPS operators cannot be (entirely) descendants. An operator annihilated by D_1 and $\bar D^4$ can be either a $(D_1)^2(\bar D^4)^2$ descendant of a long primary; or a $(D_1)^2$ or $(\bar D^4)^2$ descendant of a 1/8 BPS primary; or a 1/4 BPS primary. In [31] it was shown that a $[q,p,q]$ scalar composite operator can not be a descendant of a 1/8 BPS primary. Therefore, we argue that after subtracting off all the descendant pieces from candidate BPS operators, we should be left with a 1/4 BPS primary; it simply can not be anything else!

2.3 Examples of systematic description

We begin by outlining a few rules that determine which tensor structures are permitted. First we observe that, since contractions are made using the antisymmetric tensor ϵ^{rs} while the tensors $A_{rs\ldots t}$ are symmetric, contractions within the same A give zero,

$$A^r{}_{rs\ldots t} = 0\,, \tag{2.15}$$

so we can only contract indices in different A's.

Next, consider T^2. Since T_{rs} transforms under the 3-dimensional representation of SU(2) it follows that the product of two Ts will decompose into the five and one-dimensional representations. The former corresponds to symmetrisation on all four indices, i.e. $T_{(rs}T_{uv)}$, while for a single contraction we have

$$T_{rt}T_s{}^t = -T_{st}T_r{}^t = \frac{1}{2}\epsilon_{rs}T_{uv}T^{uv}\,. \tag{2.16}$$

Similarly, the product of three Ts contains only the seven-and three-dimensional representations, so that, for example

$$T_r{}^sT_s{}^tT_t{}^r = 0\,. \tag{2.17}$$

One can look at contractions of other A's in a similar fashion. For example, $(A_3)^2$ contains only the seven- and three-dimensional representations of SU(2) corresponding to tensors obtained by symmetrising on six or two indices with zero or two contractions respectively. So

$$A^{rst}A_{rst} = A_{(rs}{}^vA_{tu)v} = 0 \tag{2.18}$$

while

$$A_r{}^{tu}A_{stu} = A_s{}^{tu}A_{rtu} \tag{2.19}$$

or, equivalently,

$$3A_{rst}A^t{}_{uv} = \epsilon_{ru}\left(A_{stw}A^{tw}{}_v\right) + \epsilon_{rv}\left(A_{stw}A^{tw}{}_u\right) + \epsilon_{sv}\left(A_{rtw}A^{tw}{}_u\right) + \epsilon_{su}\left(A_{rtw}A^{tw}{}_v\right). \tag{2.20}$$

These equations generalise in a straightforward manner. Whenever we contract an odd number of indices in two As of the same length and symmetrize on the remaining indices, the resulting tensor vanishes.[4] If As of different length are contracted (as in $A_{rs}{}^tT_{tu}$), there is no such restriction.

We shall now discuss some explicit examples. We shall use the convention that uncontracted SU(2) indices are understood to be totally symmetrized.

The representation $[1,p,1]$**.** Such operators have to have $(p+2)$ W_r's and only one contraction. There are no single trace operators in this class because

$$\mathrm{tr}\left(W_{r_1}\ldots W_{r_p}W^2\right) = 0\,. \tag{2.21}$$

Hence these operators can only be constructed by contracting CPOs. They are all protected. This can also be seen from representation theory because there are no long representations which contain these representations [40]. This result also shows that any single-trace factor in an operator must have at least two contractions.

Here we list the lowest dimensional examples of $[1,p,1]$ representations (for $p \leq 5$). For $[1,1,1]$ and $[1,2,1]$ we can not construct any nonvanishing tensors of this form. For $[1,3,1]$, there is one possible operator,

$$\mathcal{O} = A_{rst}T^t{}_u\,. \tag{2.22}$$

[4] In general there will be more than one nonvanishing structure. For instance, both $A_{(rs}{}^{vw}A_{tu)vw}$ and $A^{rstu}A_{rstu}$ are independent nonvanishing tensors.

Similarly, for $[1,4,1]$ the only possible operator is

$$\mathcal{O} = A_{rstu} T^u{}_v \,. \tag{2.23}$$

Higher representations offer more choices, and already in the $[1,5,1]$ we find

$$\begin{aligned}
\mathcal{O}_1 &= A_{rstuv} T^v{}_w \\
\mathcal{O}_2 &= A_{rstv} A^v{}_{uw} \\
\mathcal{O}_3 &= T_{rs} T_{tv} A^v{}_{uw} \,.
\end{aligned} \tag{2.24}$$

All of these operators have protected two-point functions, as we will explicitly verify in section 3.

The representation $[2,0,2]$**.** This operator is realized as an SU(2) scalar in $(4,1,1)$ harmonic superspace. There are just two possibilities

$$\begin{aligned}
\mathcal{O}_1 &= T_{rs} T^{rs} \\
\mathcal{O}_2 &= \operatorname{tr}\left(W^2 W^2\right)
\end{aligned} \tag{2.25}$$

Using the rules outlined in the beginning of section 2.3, we see that $\mathcal{O}_1$ is the only multiple trace operator one can construct with two pairs of contracted indices, and there is also no other choice for the single trace operator but $\mathcal{O}_2$. $\mathcal{O}_1$ is a candidate 1/4 BPS operator while $\mathcal{O}_2$ is a descendant; as seen in (2.13), it is a descendant of the Konishi operator.

The representation $[2,1,2]$**.** This has 5 fields and forms an SU(2) doublet. There are again only two possibilities

$$\begin{aligned}
\mathcal{O}_1 &= A_{rst} T^{st} \\
\mathcal{O}_2 &= \operatorname{tr}\left(W_r W^2 W^2\right)
\end{aligned} \tag{2.26}$$

This case is completely parallel to the $[2,0,2]$ representation.

The operator $\mathcal{O}_2$ can be written in the form

$$\mathcal{O}_2 = -\frac{1}{16} (D_1)^2 \left(\bar{D}^4\right)^2 \operatorname{tr}\left(W_r W_{ij} \bar{W}^{ij}\right) . \tag{2.27}$$

Note that the ancestor here is defined on $(4,1,1)$ harmonic superspace; it is not G-analytic but it is H-analytic. One can easily remove the harmonic variables to obtain the corresponding superfield on ordinary superspace. In this case it is $\operatorname{tr}(W_{ij} W_{kl} \bar{W}^{kl})$.

The representation $[2,2,2]$**.** This has 6 fields and transforms as a triplet under SU(2). Multiple trace operators are constructed in the following way. We can partition the set of six fields as $6 = 4+2$, $6 = 3+3$, or $6 = 2+2+2$. Two pairs of indices are contracted, and two remaining indices are symmetrized. The possibilities are

$$\begin{aligned}
\mathcal{O}_1 &= A_{rstu} T^{tu} \\
\mathcal{O}_2 &= A_r{}^{tu} A_{stu}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{O}_3 &= T_{rs}T_{tu}T^{tu} \\
\mathcal{O}_4 &= \mathrm{tr}\left(W_r W_s W^2 W^2\right) \\
\mathcal{O}_5 &= \mathrm{tr}\left(W_r W^2 W_s W^2\right) \\
\mathcal{O}_6 &= \mathrm{tr}\left(W^2 W^2\right) T_{rs}
\end{aligned} \tag{2.28}
$$

The first three are candidate 1/4 BPS operators while the last three are descendants. For the partitions $6 = 4 + 2$ and $6 = 3 + 3$, these are the only choices because contractions within the same A give zero. For the partition $6 = 2 + 2 + 2$, equation (2.17) relates any other triple-trace $[2, 2, 2]$ operator to $\mathcal{O}_3$.

The operator $\mathcal{O}_6$ is a descendant of the product of the Konishi operator and the supercurrent, while $\mathcal{O}_4$ and $\mathcal{O}_5$ are descendants of the operators

$$
\begin{aligned}
\mathcal{A}_1 &= \mathrm{tr}\left(W_r W_s W_{ij} \bar{W}^{ij}\right) \\
\mathcal{A}_2 &= \mathrm{tr}\left(W_r W_{ij} W_s \bar{W}^{ij}\right)
\end{aligned} \tag{2.29}
$$

A short calculation yields

$$
\begin{aligned}
\mathcal{O}_4 &= \frac{1}{40}(D_1)^2 \left(\bar{D}^4\right)^2 (\mathcal{A}_2 - 3\mathcal{A}_1) \\
\mathcal{O}_5 &= \frac{1}{20}(D_1)^2 \left(\bar{D}^4\right)^2 (\mathcal{A}_1 - 2\mathcal{A}_2)\,.
\end{aligned} \tag{2.30}
$$

In terms of ordinary superfields, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are in the $[0, 2, 0]$ representation of SU(4).

The representation $[2, 3, 2]$. The possibilities are

$$
\begin{aligned}
\mathcal{O}_1 &= A_{rstuv}T^{uv} \\
\mathcal{O}_2 &= A_{rs}{}^{uv}A_{tuv} \\
\mathcal{O}_3 &= A_{rst}T_{uv}T^{uv} \\
\mathcal{O}_4 &= A_r{}^{uv}\left(T^2\right)_{stuv} \\
\mathcal{O}_5 &= \mathrm{tr}\left(W_r W_s W_t W^2 W^2\right) \\
\mathcal{O}_6 &= \mathrm{tr}\left(W_r W_s W^2 W_t W^2\right) \\
\mathcal{O}_7 &= T_{rs}\,\mathrm{tr}\left(W_t W^2 W^2\right) \\
\mathcal{O}_8 &= A_{rst}\,\mathrm{tr}\left(W^2 W^2\right)
\end{aligned} \tag{2.31}
$$

The first four are candidate 1/4 BPS operators while the second four are descendants.

The last of these is a descendant of a product of A_3 and the Konishi operator, while the ancestor of $\mathcal{O}_7$ is a product of T and $\mathrm{tr}(W_r W_{ij} \bar{W}^{ij})$. For the other two descendants we have

$$
\begin{aligned}
\mathcal{O}_5 &= \frac{1}{24}(D_1)^2 \left(\bar{D}^4\right)^2 (\mathcal{A}_2 - 2\mathcal{A}_1) \\
\mathcal{O}_6 &= \frac{1}{24}(D_1)^2 \left(\bar{D}^4\right)^2 (\mathcal{A}_1 - 2\mathcal{A}_2)
\end{aligned} \tag{2.32}
$$

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where

$$\begin{aligned}\mathcal{A}_1 &= \operatorname{tr}\left(W_r W_s W_t W_{ij} \bar{W}^{ij}\right)\\ \mathcal{A}_2 &= \operatorname{tr}\left(W_r W_s W_{ij} W_t \bar{W}^{ij}\right).\end{aligned} \tag{2.33}$$

The representation $[3,0,3]$**.** There is only one possibility:

$$\mathcal{O} = \operatorname{tr}\left(W^2 W^2 W^2\right) \tag{2.34}$$

This operator is a descendant. There are no candidate 1/4 BPS operators in this case. As we saw in (2.17) and (2.18), the operators $A_{rst}A^{rst}$ and $T_{rs}T_t{}^r T^{st}$ vanish identically. Explicitly, this operator can be written as

$$\mathcal{O} = -\frac{1}{8}(D_1)^2 \left(\bar{D}^4\right)^2 \operatorname{tr}\left(W^2 W_{ij} \bar{W}^{ij}\right). \tag{2.35}$$

The representation $[3,1,3]$**.** This example again has seven fields but the representation of SU(2) is the doublet. The operators are

$$\begin{aligned}\mathcal{O}_1 &= A_r{}^{stu} A_{stu}\\ \mathcal{O}_2 &= \left(T^2\right)_{rstu} A^{stu}\\ \mathcal{O}_3 &= \operatorname{tr}\left(W_r W^2 W^2 W^2\right)\\ \mathcal{O}_4 &= \operatorname{tr}\left(W_r W^2 W_s W^2 W^s - W_r W_s W^2 W^s W^2\right)\\ \mathcal{O}_5 &= T_r{}^s \operatorname{tr}\left(W_s W^2 W^2\right)\end{aligned} \tag{2.36}$$

so there are 3 descendants in this case. We have symmetrized $\mathcal{O}_4$ so that $\mathcal{O}_4^\dagger = +(\mathcal{O}_4)^*$. This symmetry amounts to charge conjugation on the fields X in the adjoint representation of the gauge group and the **6** of SU(4). Its effect on the $\mathcal{N}=1$ superfield formulation is to map $z_j \to z_j^t$.

The last operator is again a descendant of a product of operators that we have discussed previously. For the other two we have

$$\begin{aligned}\mathcal{O}_3 &= \frac{1}{30}(D_1)^2 \left(\bar{D}^4\right)^2 (\mathcal{A}_2 - 5\mathcal{A}_1)\\ \mathcal{O}_4 &= \frac{1}{6}(D_1)^2 \left(\bar{D}^4\right)^2 (\mathcal{A}_1 + \mathcal{A}_2)\end{aligned} \tag{2.37}$$

where

$$\begin{aligned}\mathcal{A}_1 &= \operatorname{tr}\left(W_r W^2 W_{ij} \bar{W}^{ij}\right)\\ \mathcal{A}_2 &= \operatorname{tr}\left(W_r W_s W_{ij} W^s \bar{W}^{ij}\right).\end{aligned} \tag{2.38}$$

2.4 Multiplicity of quarter BPS operators

The (classical) quarter BPS operators in the SU(4) representation are built from single trace quarter (and half) BPS operators. These have the form

$$\operatorname{tr}\left(W_{(r_1} \dots W_{r_q)} \left(W^2\right)^p\right) \tag{2.39}$$

for operators in the $[p,q,p]$ SU(4) representation, but the order of the $2p+q$ operators inside the trace is arbitrary.

To find the number of different single trace operators in this representation, N_{pq}, consider the reducible operator

$$X_Q := \mathrm{tr}\left(W_{r_1}\dots W_{r_Q}\right), \tag{2.40}$$

where the SU(2) indices are no longer taken to be symmetrised. This is in a reducible representation of SU(2), and contains all single trace scalar composite operators of dimension Q. So one obtains the number of operators in each representation by expanding this operator as a sum of irreducible representations. For example to find all single trace operators of dimension 4 consider X_4. This has 6 components (given by (1111), (1112), (1122), (1212), (2221), (2222) where $(r_1r_2r_3r_4)$ is short hand for $\mathrm{tr}(W_{r_1}\dots W_{r_4})$.) In terms of irreducible SU(2) representations it splits as $6 = 5+1$. In terms of SU(4) representations the 5 corresponds to $[0,4,0]$ and the 1 corresponds to $[2,0,2]$, and so we find that there is only one operator in each of these two representations.

More generally, to split X_Q into irreducibles, consider the components of X_Q. Let $c(Q,p)$ denote the number of components of X_Q with p 1's and $Q-p$ 2's, i.e. the number of ways to arrange a total of Q objects with p of one type and $Q-p$ of another type up to circular permutations.

Then X_Q splits into the following irreducible representations:

$$\sum_{p=0}^{\lfloor Q/2\rfloor}\left(c(Q,p)-c(Q,p-1)\right)\,[p,q,p] \tag{2.41}$$

where $q = Q-2p$ and where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x. So the number of single trace operators in the $[p,q,p]$ representation is

$$N_{pq} = c(Q,p) - c(Q,p-1). \tag{2.42}$$

In general the formula for $c(Q,p)$ is quite complicated, but in certain cases it simplifies. For example

$$c(Q,0)=1\,,\qquad c(Q,1)=1\,,\qquad c(Q,2)=\left\lfloor\frac{Q}{2}\right\rfloor, \tag{2.43}$$

and if Q and p are co-prime then

$$N_{pq} = \frac{1}{Q}\binom{Q}{p}. \tag{2.44}$$

As an example, consider dimension 6 operators: X_6 has 14 components and $c(6,p)$ is given by:

$$c(6,0)=c(6,1)=1\,,\qquad c(6,2)=3\,,\qquad c(6,3)=4\,. \tag{2.45}$$

Then (2.42) gives

$$N_{06}=1\,,\qquad N_{14}=0\,,\qquad N_{22}=2\,,\qquad N_{30}=1\,, \tag{2.46}$$

reproducing the correct numbers of single-trace operators discussed above (in particular there are two operators in the $[2,2,2]$ representation and 1 in the $[3,0,3]$.)

Since multiple trace operators can be obtained by multiplying together single trace operators, to find the number of multi-trace operators in a given representation one just has to consider all possible ways of obtaining the representation in question from tensor products of other representations and use the formula for single-trace operators.

2.5 Relationship between $\mathcal{N}=4$ and $\mathcal{N}=1$ superfields

The map between quarter BPS operators in the $\mathcal{N}=1$ formalism and those in $(4,1,1)$ analytic superspace is straightforward. In the $\mathcal{N}=1$ formalism the quarter BPS operators are given by

$$\left[\left(z^{2c}\right)^p (z_d)^q\right] \tag{2.47}$$

where $[\ldots]$ denote gauge invariant combinations, $(X_a)^p \equiv X_{(a_1}\ldots X_{a_p)}$ and $z^{2c} \equiv z_a z_b \epsilon^{abc}$. Here the $a,b,\ldots=(1,2,3)$ are SU(3) indices. These operators have highest weight state given by

$$[(z_1 z_2 - z_2 z_1)^p (z_1)^q]\,. \tag{2.48}$$

In $(4,1,1)$ harmonic superspace on the other hand, this object is given by

$$\left[\left(W^2\right)^p (W_r)^q\right]\,. \tag{2.49}$$

If we relabel the SU(2) indices $r,s,\ldots=1,2$ then this operator has highest weight state

$$[(W_1 W_2 - W_2 W_1)^p (W_1)^q]\,. \tag{2.50}$$

The correspondence between the $N=1$ operators and the harmonic superspace operators is now clear, one simply replaces W with z to obtain the highest weight states of each.

3. Explicit computations

In this section we will explicitly calculate two point functions of the above operators. We will work with the lowest components of superfields, the z_i^a and $\bar{z}_i^a$. (Here, $i=1,\ldots,3$, and a labels the adjoint representation of the gauge group SU(N).) We list operators in a given irrep of the R-symmetry group SU(4) (some of them were missed in [31]), and for the descendant operators write out the corresponding Konishi-like long operator they come from. Then we look at Born level and order g^2 contributions to the two point functions of the highest weight state operators. (Most of them were calculated in [31]).

In each representation we will have the descendant operators L_i and "candidate 1/4-BPS" operators $\mathcal{O}$. We will compute the order g^0 two point functions $\langle OL_i^\dagger\rangle_{\rm Born}$ and $\langle L_i L_j^\dagger\rangle_{\rm Born}$. Then we will consider operators

$$\tilde{\mathcal{O}} \equiv \mathcal{O} - \left\langle OL_i^\dagger\right\rangle_{\rm Born}\left(\left\langle LL^\dagger\right\rangle_{\rm Born}^{-1}\right)^{ij} L_j\,. \tag{3.1}$$

By construction, they are orthogonal to all the L_i at Born level, $\langle\tilde{\mathcal{O}}L_i^\dagger\rangle_{\rm Born}=0$. Then we will show that these operators $\tilde{\mathcal{O}}$ have protected two point functions at order g^2, $\langle\tilde{\mathcal{O}}L_i^\dagger\rangle_{g^2}=0$ and $\langle\tilde{\mathcal{O}}'\tilde{\mathcal{O}}^\dagger\rangle_{g^2}=0$ for all such operators $\tilde{\mathcal{O}},\tilde{\mathcal{O}}'$. The claim is that these operators $\tilde{\mathcal{O}}$ are 1/4-BPS.

The basis of operators we will choose is slightly different from the one used in [31]. The operators introduced in the preceding section are more natural and intuitive.

The representation $[1,p,1]$**.**

- There is only one operator in the representation $[1,3,1]$ whose highest SU(4) weight state is

$$\mathcal{O} \equiv \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_2 - \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_1 \tag{3.2}$$

while acting on $\mathcal{O}$ once with an SU(4) ladder operator gives

$$\mathcal{O}' \equiv 2 \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_2 z_2 - \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_2 - \operatorname{tr} z_2 z_2 \operatorname{tr} z_1 z_1 z_1 \,. \tag{3.3}$$

This operator has the same weight as the $[2,1,2]$ operators (but is of course orthogonal to them). The Born and order g^2 overlaps are

$$\langle \mathcal{O}' \mathcal{O}'^{\dagger} \rangle_{\text{Born}} = \frac{15}{32} N \left(N^2-1\right)\left(N^2-4\right), \qquad \langle \mathcal{O}' \mathcal{O}'^{\dagger} \rangle_{g^2} = 0 \,. \tag{3.4}$$

So indeed it is a 1/4-BPS operator.

- There is only one operator in the representation $[1,4,1]$. The highest SU(4) weight state operator is

$$\mathcal{O} \equiv \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_1 z_2 - \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_1 z_1 \tag{3.5}$$

while acting on $\mathcal{O}$ once with an SU(4) ladder operator gives

$$\mathcal{O}' \equiv 2 \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_2 z_2 + \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_2 z_1 z_2 - 2 \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_1 z_2 - \operatorname{tr} z_2 z_2 \operatorname{tr} z_1 z_1 z_1 z_1 \,. \tag{3.6}$$

This operator has the same weight as the $[2,2,2]$ operators (but is of course orthogonal to them). The Born and order g^2 overlaps are

$$\langle \mathcal{O}' \mathcal{O}'^{\dagger} \rangle_{\text{Born}} = \frac{3}{8} (N^2-1)(N^2-4)(N^2-9) \,, \qquad \langle \mathcal{O}' \mathcal{O}'^{\dagger} \rangle_{g^2} = 0 \,. \tag{3.7}$$

So indeed it is a 1/4-BPS operator.

- Finally, there are 3 operators in the representation $[1,5,1]$. Their highest SU(4) weight state operators are

$$\mathcal{O}_1 \equiv \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_1 z_1 z_2 - \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_1 z_1 z_1 \tag{3.8}$$

$$\mathcal{O}_2 \equiv \operatorname{tr} z_1 z_1 z_1 \operatorname{tr} z_1 z_1 z_1 z_2 - \operatorname{tr} z_1 z_1 z_2 \operatorname{tr} z_1 z_1 z_1 z_1 \tag{3.9}$$

$$\mathcal{O}_3 \equiv \operatorname{tr} z_1 z_1 \left(\operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_2 - \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_1\right) \tag{3.10}$$

while acting on $\mathcal{O}$ once with an SU(4) ladder operator gives

$$\begin{aligned} \mathcal{O}_1' \equiv\ & 2 \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_1 z_2 z_2 + 2 \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_2 z_1 z_2 - \\ & - 3 \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_1 z_1 z_2 - \operatorname{tr} z_2 z_2 \operatorname{tr} z_1 z_1 z_1 z_1 z_1 \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathcal{O}_2' \equiv\ & 2 \operatorname{tr} z_1 z_1 z_1 \operatorname{tr} z_1 z_1 z_2 z_2 + \operatorname{tr} z_1 z_1 z_1 \operatorname{tr} z_1 z_2 z_1 z_2 - \\ & - \operatorname{tr} z_1 z_1 z_2 \operatorname{tr} z_1 z_1 z_1 z_2 - 2 \operatorname{tr} z_1 z_2 z_2 \operatorname{tr} z_1 z_1 z_1 z_1 \end{aligned} \tag{3.12}$$

$$\begin{aligned} \mathcal{O}_3' \equiv\ & 2 \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_2 z_2 + \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_2 - \\ & - 2 \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_2 \operatorname{tr} z_1 z_1 z_1 - \operatorname{tr} z_2 z_2 \operatorname{tr} z_1 z_1 \operatorname{tr} z_1 z_1 z_1 \,. \end{aligned} \tag{3.13}$$

These operators have the same weight as the $[2,3,2]$ operators (but are of course orthogonal to them). The Born and order g^2 overlaps are

$$\langle \mathcal{O}'_i \mathcal{O}'^{\dagger}_j \rangle_{\text{Born}} = \tfrac{35(N^2-1)(N^2-4)}{128N} \times$$
$$\times \begin{pmatrix} N^4 - 10N^2 + 72 & -11N^2 + 36 & 6N(N^2-2) \\ & N^4 - 4N^2 + 18 & -2N(2N^2+3) \\ & & 2N^2(N^2+5) \end{pmatrix}, \quad (3.14)$$

$$\langle \mathcal{O}'_i \mathcal{O}'^{\dagger}_j \rangle_{g^2} = 0\,. \quad (3.15)$$

So indeed they all are 1/4-BPS operators.

The representation $[2,0,2]$**.** The operators corresponding to (2.25) are

$$\begin{aligned} \mathcal{O}_1 &= 2\,(\mathrm{tr}\, z_1 z_1\, \mathrm{tr}\, z_2 z_2 - \mathrm{tr}\, z_1 z_2\, \mathrm{tr}\, z_1 z_2) \\ \mathcal{O}_2 &= \mathrm{tr}\, z_1 z_1 z_2 z_2 - \mathrm{tr}\, z_1 z_2 z_1 z_2\,. \end{aligned} \quad (3.16)$$

The single trace operator $\mathcal{O}_2$ is a descendant of the Konishi scalar,

$$\begin{aligned} \left(\bar{Q}_{\bar\zeta}\right)^2 \mathrm{tr}\, z_j \bar{z}^j &= \bar{Q}_{\bar\zeta}\, \mathrm{tr}\, z_j \sqrt{2}\bar\zeta\bar\psi^j = 6i\sqrt{2}(\bar\zeta\bar\zeta)\, \mathrm{tr}[z_1,z_2]z_3, \\ \left(Q_{\zeta_3}\right)^2 \left(\bar{Q}_{\bar\zeta}\right)^2 \mathrm{tr}\, z_j \bar{z}^j &= -12i\left(\bar\zeta\bar\zeta\right) Q_{\zeta_3}\, \mathrm{tr}[z_1,z_2]\zeta_3\lambda \\ &= 24\left(\bar\zeta\bar\zeta\right)\left(\zeta_3\zeta_3\right) \mathrm{tr}[z_1,z_2]^2 \\ &= -48\left(\bar\zeta\bar\zeta\right)\left(\zeta_3\zeta_3\right)\mathcal{O}_2\,; \end{aligned} \quad (3.17)$$

or

$$\mathcal{O}_2 \sim \left(Q^2\bar{Q}^2\right) \mathrm{tr}\, z_j \bar{z}^j\,. \quad (3.18)$$

for short.[5] On the other hand, the operator orthogonal to $\mathcal{O}_2$ at Born level

$$\tilde{\mathcal{O}}_1 = \mathcal{O}_1 - \frac{4}{N}\mathcal{O}_2 \quad (3.19)$$

stays orthogonal to $\mathcal{O}_2$, $\langle \tilde{\mathcal{O}}_1(x)\bar{\mathcal{O}}_2(y)\rangle = 0$; and has a two-point function $\langle \tilde{\mathcal{O}}_1(x)\bar{\tilde{\mathcal{O}}}_1(y)\rangle = \langle \tilde{\mathcal{O}}_1(x)\bar{\tilde{\mathcal{O}}}_1(y)\rangle_{\text{Born}}$ protected at order g^2.

The representation $[2,1,2]$**.** In this representation we again have only two operators

$$\begin{aligned} \mathcal{O}_1 &= \mathrm{tr}\, z_1 z_1 z_1\, \mathrm{tr}\, z_2 z_2 - 2\,\mathrm{tr}\, z_1 z_2\, \mathrm{tr}\, z_1 z_2 z_1 + \mathrm{tr}\, z_1 z_2 z_2\, \mathrm{tr}\, z_1 z_1 \\ \mathcal{O}_2 &= \mathrm{tr}\, z_1 z_1 z_1 z_2 z_2 - \mathrm{tr}\, z_1 z_1 z_2 z_1 z_2 \end{aligned} \quad (3.20)$$

and the single trace operator is again a descendant,

$$\mathcal{O}_2 \sim (Q^2\bar{Q}^2)\, \mathrm{tr}\left[z_1 z_j \bar{z}^j + z_1 \bar{z}^j z_j\right]. \quad (3.21)$$

The operator orthogonal to $\mathcal{O}_2$ at Born level

$$\tilde{\mathcal{O}}_1 = \mathcal{O}_1 - \frac{6}{N}\mathcal{O}_2 \quad (3.22)$$

satisfies $\langle \tilde{\mathcal{O}}_1(x)\bar{\mathcal{O}}_2(y)\rangle = 0$, $\langle \tilde{\mathcal{O}}_1(x)\bar{\tilde{\mathcal{O}}}_1(y)\rangle = \langle \tilde{\mathcal{O}}_1(x)\bar{\tilde{\mathcal{O}}}_1(y)\rangle_{\text{Born}}$ at order g^2.

[5] We will not write out the indices of the supercharges or proportionality constant explicitly from now on. The supercharges will be always the same as in (3.17), and keeping track of all the factors like 48 or $(\bar\zeta\bar\zeta)(\zeta_3\zeta_3)$ would only clutter the notation.

The representation $[2,2,2]$. Here we have a total of six operators, one of which was missed in [31]. The lowest components of superfields (2.28) are

$$\begin{aligned}
\mathcal{O}_1 &\equiv 3\,\mathrm{tr}\, z_1z_1z_1z_1\,\mathrm{tr}\, z_2z_2 - 6\,\mathrm{tr}\, z_1z_1z_1z_2\,\mathrm{tr}\, z_1z_2 + (2\,\mathrm{tr}\, z_1z_1z_2z_2 + \mathrm{tr}\, z_1z_2z_1z_2)\,\mathrm{tr}\, z_1z_1\\
\mathcal{O}_2 &\equiv \mathrm{tr}\, z_1z_1z_1\,\mathrm{tr}\, z_1z_2z_2 - \mathrm{tr}\, z_1z_1z_2\,\mathrm{tr}\, z_1z_1z_2\\
\mathcal{O}_3 &\equiv \mathrm{tr}\, z_1z_1\,(\mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_2z_2 - \mathrm{tr}\, z_1z_2\,\mathrm{tr}\, z_1z_2)\\
\mathcal{O}_4 &\equiv \mathrm{tr}\, z_1z_1z_1z_1z_2z_2 - 2\,\mathrm{tr}\, z_1z_1z_1z_2z_1z_2 + \mathrm{tr}\, z_1z_1z_2z_1z_1z_2\\
\mathcal{O}_5 &\equiv \mathrm{tr}\, z_1z_1z_1z_2z_1z_2 - \mathrm{tr}\, z_1z_1z_2z_1z_1z_2\\
\mathcal{O}_6 &\equiv \mathrm{tr}\, z_1z_1\,(\mathrm{tr}\, z_1z_1z_2z_2 - \mathrm{tr}\, z_1z_2z_1z_2)
\end{aligned} \tag{3.23}$$

(and we didn't bother to keep the same normalization factors for all of them — just whatever looks better). The descendants arise from the Konishi-like long primary operators as

$$\begin{aligned}
\mathcal{O}_4 &\sim \left(Q^2\bar{Q}^2\right)\mathrm{tr}\left[z_1z_1z_j\bar{z}^j + z_1z_1\bar{z}^jz_j\right]\\
\mathcal{O}_5 &\sim \left(Q^2\bar{Q}^2\right)\mathrm{tr}\left[z_1z_jz_1\bar{z}^j\right]\\
\mathcal{O}_6 &\sim \left(Q^2\bar{Q}^2\right)\left[\mathrm{tr}\, z_1z_1\right]\left[\mathrm{tr}\, z_j\bar{z}^j\right].
\end{aligned} \tag{3.24}$$

Note that another operator exists in a long multiplet, whose descendant coincides with $\mathcal{O}_6$,

$$\mathcal{O}_6 \sim \left(Q^2\bar{Q}^2\right)\left[\mathrm{tr}\, z_1z_j\right]\left[\mathrm{tr}\, z_1\bar{z}^j\right]. \tag{3.25}$$

This may be established by observing that the difference operator,

$$3\left[\mathrm{tr}\, z_1z_j\right]\left[\mathrm{tr}\, z_1\bar{z}^j\right] - \left[\mathrm{tr}\, z_1z_1\right]\left[\mathrm{tr}\, z_j\bar{z}^j\right] \tag{3.26}$$

is semi-short.[6] A similar phenomenon occurs for higher representations, and we will not mention it explicitly.

The linear combinations orthogonal to these operators at Born level can be taken as

$$\begin{aligned}
\tilde{\mathcal{O}}_1 &= \mathcal{O}_1 - \frac{24}{N}\mathcal{O}_4 - \frac{48(2N^2-3)}{N(3N^2-2)}\mathcal{O}_5 + \frac{40}{3N^2-2}\mathcal{O}_6\\
\tilde{\mathcal{O}}_2 &= \mathcal{O}_2 - \frac{4}{N}\mathcal{O}_4 - \frac{3(7N^2-8)}{N(3N^2-2)}\mathcal{O}_5 + \frac{5}{3N^2-2}\mathcal{O}_6\\
\tilde{\mathcal{O}}_3 &= \mathcal{O}_3 - \frac{20}{3N^2-2}\mathcal{O}_5 - \frac{10}{3N^2-2}\mathcal{O}_6\,.
\end{aligned} \tag{3.27}$$

The matrix of two point functions in this basis is

$$\begin{pmatrix} \langle\tilde{\mathcal{O}}_i\tilde{\mathcal{O}}_j^\dagger\rangle_{\mathrm{Born}} & 0\\ 0 & \langle L_iL_j^\dagger\rangle_{\mathrm{Born}}\end{pmatrix} + \begin{pmatrix} 0 & 0\\ 0 & \langle L_iL_j^\dagger\rangle_{g^2}\end{pmatrix} \tag{3.28}$$

where the (symmetric) blocks are

$$\langle L_iL_j^\dagger\rangle_{\mathrm{Born}} = \tfrac{(N^2-1)}{64}\begin{pmatrix} 7N^4+20N^2+8 & -4N^4-10N^2+4 & 2N(13N^2-2)\\ & 3N^4+2 & -2N(6N^2+1)\\ & & 2N^2(3N^2+13)\end{pmatrix} \tag{3.29}$$

[6] This is the non-renormalised 20' operator discussed in [21, 41].

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and

$$\langle L_i L_j^\dagger\rangle_{g^2} = \tfrac{3\tilde{B}N(N^2-1)}{32}\begin{pmatrix} 25N^4+148N^2+8 & 15N^4-66N^2+4 & 4N(27N^2+17) \\ & 10N^4+22N^2+2 & -2N(28N^2+13) \\ & & 2N^2(9N^2+89) \end{pmatrix} \tag{3.30}$$

and

$$\begin{aligned}
\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_1^\dagger\rangle_{\rm Born} &= \frac{360\mathcal{C}_N}{N^2(3N^2-2)}\times\left(N^6-11N^4+70N^2-48\right) \\
\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2^\dagger\rangle_{\rm Born} &= \frac{-120\mathcal{C}_N}{N^2(3N^2-2)}\times\left(5N^4-36N^2+24\right) \\
\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_3^\dagger\rangle_{\rm Born} &= \frac{240\mathcal{C}_N}{N(3N^2-2)}\times\left(N^2-2\right)\left(2N^2-3\right) \\
\langle\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_2^\dagger\rangle_{\rm Born} &= \frac{5\mathcal{C}_N}{N^2(3N^2-2)}\times\left(3N^6-41N^4+160N^2-96\right) \\
\langle\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^\dagger\rangle_{\rm Born} &= \frac{-20\mathcal{C}_N}{N(3N^2-2)}\times\left(13N^2-12\right) \\
\langle\tilde{\mathcal{O}}_3\tilde{\mathcal{O}}_3^\dagger\rangle_{\rm Born} &= \frac{60\ \mathcal{C}_N}{(3N^2-2)}\times\left(N^2+1\right)\left(N^2-2\right).
\end{aligned} \tag{3.31}$$

Here and below we shall use the abbreviation,

$$\mathcal{C}_N \equiv \frac{(N^2-1)(N^2-4)}{64}. \tag{3.32}$$

As seen from (3.28), the operators defined in (3.27) have protected two-point functions $\langle\tilde{\mathcal{O}}_i\tilde{\mathcal{O}}_j^\dagger\rangle$ at order g^2. This shows that we can argue that $\tilde{\mathcal{O}}_1$, $\tilde{\mathcal{O}}_2$, $\tilde{\mathcal{O}}_3$ are the 1/4-BPS primaries we are after. Anomalous scaling dimensions of long operators ($L_i = \mathcal{O}_4, \mathcal{O}_5, \mathcal{O}_6$) match those of their Konishi-like primaries computed in [41].

A better basis for protected $[2,2,2]$ operators. It may seem odd that the operators mixing of the operators in the representations $[2,0,2]$ and $[2,1,2]$ are in terms of coefficients that are merely inverse powers of N, while the mixing coefficients for the operators we identified in the representation $[2,2,2]$ have more complicated denominators. This distinction would also be surprising from the perspective of AdS/CFT, since the more complicated denominators would suggest that an infinite series of corrections in the string coupling $g_s = \lambda/N$ would appear for given 't Hooft coupling λ. As a matter of fact, the mixing coefficients depend upon the bases chosen for both the $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ operators as well as the pure descendants. In a different basis, the coefficients are all proportional to inverse powers of N. For the representation $[2,2,2]$, these new operators are found easily, and we have

$$\begin{aligned}
\tilde{\mathcal{O}}_1' &= \tilde{\mathcal{O}}_1 + \frac{4}{N}\tilde{\mathcal{O}}_3 = \mathcal{O}_1 + \frac{4}{N}\mathcal{O}_3 - \frac{24}{N}\mathcal{O}_4 - \frac{32}{N}\mathcal{O}_5 \\
\tilde{\mathcal{O}}_2' &= \tilde{\mathcal{O}}_2 + \frac{1}{2N}\tilde{\mathcal{O}}_3 = \mathcal{O}_2 - \frac{4}{N}\mathcal{O}_4 - \frac{7}{N}\mathcal{O}_5 \\
\tilde{\mathcal{O}}_3' &= \tilde{\mathcal{O}}_3 + \frac{2}{3N}\tilde{\mathcal{O}}_1 - \frac{4}{N}\tilde{\mathcal{O}}_2 = \mathcal{O}_3 + \frac{2}{3N}\mathcal{O}_1 - \frac{4}{N}\mathcal{O}_2 - \frac{10}{3N}\mathcal{O}_6\,.
\end{aligned} \tag{3.33}$$

In this new basis, the matrix of 2-pt functions now reads

$$\langle \tilde{\mathcal{O}}'_i \tilde{\mathcal{O}}'^{\dagger}_j \rangle_{\rm Born} = \frac{5\mathcal{C}_N}{N^2} \begin{pmatrix} 24(N^4+3N^2+32) & -16(N^2+9) & 16N(4N^2-1) \\ & (N^2-3)(N^2+9) & -2N(N^2-9) \\ & & 4(N^4+17N^2-24) \end{pmatrix} \tag{3.34}$$

The representation $[2,3,2]$**.** Here there is a total of 8 operators (one was overlooked in [31]). The operators corresponding to the basis (2.31) are

$$\begin{aligned}
\mathcal{O}_1 &\equiv 2\,\mathrm{tr}\, z_1z_1z_1z_1z_1\,\mathrm{tr}\, z_2z_2 - 4\,\mathrm{tr}\, z_1z_1z_1z_1z_2\,\mathrm{tr}\, z_1z_2 + (\mathrm{tr}\, z_1z_1z_1z_2z_2 + \mathrm{tr}\, z_1z_1z_2z_1z_2)\,\mathrm{tr}\, z_1z_1 \\
\mathcal{O}_2 &\equiv 3\,\mathrm{tr}\, z_1z_1z_1z_1\,\mathrm{tr}\, z_1z_2z_2 - 6\,\mathrm{tr}\, z_1z_1z_1z_2\,\mathrm{tr}\, z_1z_1z_2 + (2\,\mathrm{tr}\, z_1z_1z_2z_2 + \mathrm{tr}\, z_1z_2z_1z_2)\,\mathrm{tr}\, z_1z_1z_1 \\
\mathcal{O}_3 &\equiv \mathrm{tr}\, z_1z_1z_1\,(\mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_2z_2 - \mathrm{tr}\, z_1z_2\,\mathrm{tr}\, z_1z_2) \\
\mathcal{O}_4 &\equiv \mathrm{tr}\, z_1z_1z_1\,\mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_2z_2 - 2\,\mathrm{tr}\, z_1z_1z_2\,\mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_1z_2 + \mathrm{tr}\, z_1z_2z_2\,\mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_1z_1 \\
\mathcal{O}_5 &\equiv \mathrm{tr}\, z_1z_1z_1z_1z_1z_2z_2 - 2\,\mathrm{tr}\, z_1z_1z_1z_1z_2z_1z_2 + \mathrm{tr}\, z_1z_1z_1z_2z_1z_1z_2 \\
\mathcal{O}_6 &\equiv \mathrm{tr}\, z_1z_1z_1z_1z_2z_1z_2 - \mathrm{tr}\, z_1z_1z_1z_2z_1z_1z_2 \\
\mathcal{O}_7 &\equiv (\mathrm{tr}\, z_1z_1z_1z_2z_2 - \mathrm{tr}\, z_1z_1z_2z_1z_2)\,\mathrm{tr}\, z_1z_1 \\
\mathcal{O}_8 &\equiv (\mathrm{tr}\, z_1z_1z_2z_2 - \mathrm{tr}\, z_1z_2z_1z_2)\,\mathrm{tr}\, z_1z_1z_1\,.
\end{aligned} \tag{3.35}$$

Out of these, four are descendants,

$$\begin{aligned}
\mathcal{O}_5 &\sim \left(Q^2\bar{Q}^2\right)\,\mathrm{tr}\left[2z_1z_1z_1z_j\bar{z}^j + 2z_1z_1z_1\bar{z}^jz_j - z_1z_1z_jz_1\bar{z}^j - z_1z_1\bar{z}^jz_1z_j\right] \\
\mathcal{O}_6 &\sim \left(Q^2\bar{Q}^2\right)\,\mathrm{tr}\left[z_1z_1z_1z_j\bar{z}^j + z_1z_1z_1\bar{z}^jz_j - 2z_1z_1z_jz_1\bar{z}^j - 2z_1z_1\bar{z}^jz_1z_j\right] \\
\mathcal{O}_7 &\sim \left(Q^2\bar{Q}^2\right)\left[\mathrm{tr}\, z_1z_1z_j\,\mathrm{tr}\, z_1\bar{z}^j + \mathrm{tr}\, z_1z_1\bar{z}^j\,\mathrm{tr}\, z_1z_j - 12\,\mathrm{tr}\, z_1z_1z_1\,\mathrm{tr}\, z_j\bar{z}^j\right] \\
\mathcal{O}_8 &\sim \left(Q^2\bar{Q}^2\right)\left[\mathrm{tr}\, z_1z_1z_1\,\mathrm{tr}\, z_j\bar{z}^j\right]
\end{aligned} \tag{3.36}$$

while the combinations orthogonal to them at Born level can be taken as

$$\begin{aligned}
\tilde{\mathcal{O}}_1 &= \mathcal{O}_1 - \frac{20}{N}\mathcal{O}_5 - \frac{30(N^2-2)}{N^3}\mathcal{O}_6 + \frac{15(N^2-2)}{N^4}\mathcal{O}_7 + \frac{10(N^2+2)}{N^4}\mathcal{O}_8 \\
\tilde{\mathcal{O}}_2 &= \mathcal{O}_2 - \frac{30}{N}\mathcal{O}_5 - \frac{30(2N^2-3)}{N^3}\mathcal{O}_6 + \frac{15(2N^2-3)}{N^4}\mathcal{O}_7 + \frac{10(N^2+3)}{N^4}\mathcal{O}_8 \\
\tilde{\mathcal{O}}_3 &= \mathcal{O}_3 - \frac{12}{N^2}\mathcal{O}_6 - \frac{3(N^2-2)}{N^3}\mathcal{O}_7 - \frac{2(N^2+2)}{N^3}\mathcal{O}_8 \\
\tilde{\mathcal{O}}_4 &= \mathcal{O}_4 - \frac{18}{N^2}\mathcal{O}_6 - \frac{(7N^2-9)}{N^3}\mathcal{O}_7 - \frac{2(2N^2+9)}{3N^3}\mathcal{O}_8\,.
\end{aligned} \tag{3.37}$$

The matrix of two point functions in this basis is

$$\begin{pmatrix} \langle \tilde{\mathcal{O}}_i \tilde{\mathcal{O}}^{\dagger}_j \rangle_{\rm Born} & 0 \\ 0 & \langle L_i L^{\dagger}_j \rangle_{\rm Born} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \langle L_i L^{\dagger}_j \rangle_{g^2} \end{pmatrix} \tag{3.38}$$

so indeed at order g^2 the operators defined in (3.37) have protected correlators. This shows that we can argue that $\tilde{\mathcal{O}}_1$, $\tilde{\mathcal{O}}_2$, $\tilde{\mathcal{O}}_3$, $\tilde{\mathcal{O}}_4$ are the 1/4-BPS primaries we are after. The

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(symmetric) blocks in equation (3.38) are

$$\langle L_i L_j^\dagger\rangle_{\text{Born}} = \tfrac{1}{2} N\mathcal{C}_N \begin{pmatrix} 6N^2+45 & -3N^2-9 & 18N & 36N \\ & 2N^2-3 & -4N & -15N \\ & & 4N^2+24 & 36 \\ & & & 9N^2+54 \end{pmatrix} \tag{3.39}$$

and

$$\langle L_i L_j^\dagger\rangle_{g^2} = 12\tilde{B}N\mathcal{C}_N \times \begin{pmatrix} 9N(2N^2+27) & -9N(N^2+8) & 54(N^2+2) & 27(5N^2+6) \\ & N(5N^2+17) & -6(3N^2+2) & -6(10N^2+3) \\ & & 4N(2N^2+23) & 174N \\ & & & 9N(3N^2+35) \end{pmatrix}. \tag{3.40}$$

The Born level overlaps of protected operators are given by ugly and not particularly illuminating expressions. Here we list them for the sake of completeness:

$$\begin{aligned}
\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_1^\dagger\rangle_{\text{Born}} &= 30\mathcal{C}_N N^{-5}\left(N^8-10N^6+117N^4-720N^2+420\right)\\
\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_2^\dagger\rangle_{\text{Born}} &= -90\mathcal{C}_N N^{-5}\left(4N^6-69N^4+395N^2-210\right)\\
\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_3^\dagger\rangle_{\text{Born}} &= 90\mathcal{C}_N N^{-4}\left(N^2-2\right)\left(N^4-7N^2+14\right)\\
\langle\tilde{\mathcal{O}}_1\tilde{\mathcal{O}}_4^\dagger\rangle_{\text{Born}} &= 30\mathcal{C}_N N^{-4}\left(N^2-2\right)\left(N^2-9\right)\left(3N^2-7\right)\\
\langle\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_2^\dagger\rangle_{\text{Born}} &= 45\mathcal{C}_N N^{-5}\left(N^8-29N^6+328N^4-1290N^2+630\right)\\
\langle\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_3^\dagger\rangle_{\text{Born}} &= -630\mathcal{C}_N N^{-4}\left(N^2-1\right)\left(N^2-6\right)\\
\langle\tilde{\mathcal{O}}_2\tilde{\mathcal{O}}_4^\dagger\rangle_{\text{Born}} &= 30\mathcal{C}_N N^{-4}\left(N^2-1\right)\left(N^2-9\right)\left(2N^2-21\right)\\
\langle\tilde{\mathcal{O}}_3\tilde{\mathcal{O}}_3^\dagger\rangle_{\text{Born}} &= 9\mathcal{C}_N N^{-3}\left(N^6-N^4-16N^2+56\right)\\
\langle\tilde{\mathcal{O}}_3\tilde{\mathcal{O}}_4^\dagger\rangle_{\text{Born}} &= 6\mathcal{C}_N N^{-3}\left(N^2-9\right)\left(N^4+3N^2-14\right)\\
\langle\tilde{\mathcal{O}}_4\tilde{\mathcal{O}}_4^\dagger\rangle_{\text{Born}} &= 2\mathcal{C}_N N^{-3}\left(N^2-9\right)\left(7N^4+16N^2-63\right)
\end{aligned} \tag{3.41}$$

and the constant $\mathcal{C}_N = (N^2-1)(N^2-4)/64$ was defined in (3.32).

The representation $[3,1,3]$**.** Here there is a total of 5 operators (one was overlooked in [31]). The highest weight states lowest component operators corresponding to the basis (2.36) are

$$\begin{aligned}
\mathcal{O}_1 \equiv\; & \operatorname{tr} z_1z_1z_1z_1 \operatorname{tr} z_2z_2z_2 - 3\operatorname{tr} z_1z_1z_1z_2 \operatorname{tr} z_1z_2z_2 + \\
& + (2\operatorname{tr} z_1z_1z_2z_2 + \operatorname{tr} z_1z_2z_1z_2)\operatorname{tr} z_1z_1z_2 - \operatorname{tr} z_1z_2z_2z_2 \operatorname{tr} z_1z_1z_1\\
\mathcal{O}_2 \equiv\; & \operatorname{tr} z_1z_2\,(2\operatorname{tr} z_1z_2 \operatorname{tr} z_1z_2z_2 - \operatorname{tr} z_2z_2 \operatorname{tr} z_1z_1z_1 - 3\operatorname{tr} z_1z_1 \operatorname{tr} z_1z_2z_2) + \\
& + \operatorname{tr} z_1z_1\,(\operatorname{tr} z_2z_2 \operatorname{tr} z_1z_1z_2 + \operatorname{tr} z_1z_1 \operatorname{tr} z_2z_2z_2)\\
\mathcal{O}_3 \equiv\; & \operatorname{tr} z_1z_1z_1z_2z_1z_2z_2 - \operatorname{tr} z_1z_1z_1z_2z_2z_1z_2\\
\mathcal{O}_4 \equiv\; & 2\operatorname{tr} z_1z_1z_1z_1z_2z_2z_2 - 3\operatorname{tr} z_1z_1z_1z_2z_1z_2z_2 - 3\operatorname{tr} z_1z_1z_1z_2z_2z_1z_2 + \\
& + 2\operatorname{tr} z_1z_1z_2z_1z_2z_1z_2 + 2\operatorname{tr} z_1z_1z_2z_2z_1z_1z_2\\
\mathcal{O}_5 \equiv\; & -(\operatorname{tr} z_1z_1z_2z_2z_2 - \operatorname{tr} z_1z_2z_1z_2z_2)\operatorname{tr} z_1z_1 + \\
& + (\operatorname{tr} z_1z_1z_1z_2z_2 - \operatorname{tr} z_1z_1z_2z_1z_2)\operatorname{tr} z_1z_2\,.
\end{aligned} \tag{3.42}$$

Out of these three are descendants,

$$
\begin{aligned}
\mathcal{O}_3 &\sim \left(Q^2\bar{Q}^2\right) \mathrm{tr}\Big[z_1 z_1 z_2 z_j \bar{z}^j + z_1 z_1 z_2 \bar{z}^j z_j - z_1 z_1 z_j \bar{z}^j z_2 - z_1 z_1 \bar{z}^j z_j z_2\Big] \\
\mathcal{O}_4 &\sim \left(Q^2\bar{Q}^2\right) \mathrm{tr}\Big[z_1 z_1 z_2 z_j \bar{z}^j + z_1 z_1 z_2 \bar{z}^j z_j + z_1 z_1 z_j \bar{z}^j z_2 + z_1 z_1 \bar{z}^j z_j z_2 - \\
&\qquad\qquad - 2 z_1 z_2 z_1 z_j \bar{z}^j - 2 z_1 z_2 z_1 \bar{z}^j z_j\Big] \qquad (3.43) \\
\mathcal{O}_5 &\sim \left(Q^2\bar{Q}^2\right) \Big[2\,\mathrm{tr}\, z_1 z_1 z_j\, \mathrm{tr}\, z_2 \bar{z}^j + 2\,\mathrm{tr}\, z_1 z_1 \bar{z}^j\, \mathrm{tr}\, z_2 z_j - \mathrm{tr}\, z_1 z_2 \bar{z}^j\, \mathrm{tr}\, z_1 z_j - \\
&\qquad - \mathrm{tr}\, z_1 z_2 z_j\, \mathrm{tr}\, z_1 \bar{z}^j - \mathrm{tr}\, z_2 z_1 \bar{z}^j\, \mathrm{tr}\, z_1 z_j - \mathrm{tr}\, z_2 z_1 z_j\, \mathrm{tr}\, z_1 \bar{z}^j\Big]
\end{aligned}
$$

while the combinations orthogonal to them at Born level can be taken as

$$
\begin{aligned}
\tilde{\mathcal{O}}_1 &= \mathcal{O}_1 - \frac{2N}{N^2-2}\mathcal{O}_4 - \frac{5}{N^2-2}\mathcal{O}_5 \\
\tilde{\mathcal{O}}_2 &= \mathcal{O}_2 + \frac{8}{N^2-2}\mathcal{O}_4 + \frac{10N}{N^2-2}\mathcal{O}_5 \,. \qquad (3.44)
\end{aligned}
$$

The matrix of two point functions in this basis is

$$
\begin{pmatrix} \langle \tilde{\mathcal{O}}_i \tilde{\mathcal{O}}_j^\dagger \rangle_{\mathrm{Born}} & 0 \\ 0 & \langle L_i L_j^\dagger \rangle_{\mathrm{Born}} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \langle L_i L_j^\dagger \rangle_{g^2} \end{pmatrix} \qquad (3.45)
$$

so indeed at order g^2 the operators defined in (3.44) have protected correlators. This shows that we can argue that $\tilde{\mathcal{O}}_1$, $\tilde{\mathcal{O}}_2$, are the 1/4-BPS primaries we are after. The (symmetric) blocks in (3.45) are

$$
\begin{aligned}
\langle L_i L_j^\dagger \rangle_{\mathrm{Born}} &= \mathcal{C}_N \begin{pmatrix} N(N^2-9) & 0 & 0 \\ & 15N(N^2+3) & -30N^2 \\ & & 3N(N^2+6) \end{pmatrix} \\
\langle L_i L_j^\dagger \rangle_{g^2} &= 6\tilde{B}N\mathcal{C}_N \begin{pmatrix} 5N(N^4-9) & 0 & 0 \\ & 75N(N^2+7) & 180(N^2+1) \\ & & 12N(N^2+16) \end{pmatrix} \\
\langle \tilde{\mathcal{O}}_i \tilde{\mathcal{O}}_j^\dagger \rangle_{\mathrm{Born}} &= \frac{15(N^2-9)}{N^2(N^2-2)}\, \mathcal{C}_N \begin{pmatrix} (N^2-1)(N^2-4) & 4N(N^2-1) \\ & 2N^2(N^2-6) \end{pmatrix} . \qquad (3.46)
\end{aligned}
$$

Here we should mention that the operator $\mathcal{O}_3$ (which was overlooked in [31]) has zero correlators with everything else. The reason is that $\mathcal{O}_3^\dagger = -(\mathcal{O}_3)^*$ while all other operators satisfy $\mathcal{O}_i^\dagger = +(\mathcal{O}_i)^*$.

Completeness of the construction. An important point which remains to be addressed is whether the construction of the 1/4 BPS operators given above is exhaustive. The fact that it is follows from SU(4) group theory in the following manner. Given a 1/4 BPS representation of SU(4) of the type $[q,p,q]$, one begins by listing all possible monomial scalar composite operators built out of $(p+q)$ z_1's and q z_2's.[7] These monomials form a

[7] The remaining scalar field z_3 never enters in the highest weight of a 1/4 BPS representation, though it will also be needed when describing 1/8 BPS operators.

basis for the linear space of scalar composite operators of the form $[(z_1)^{(p+q)}\,(z_2)^q]$. They can occur in representations $[0,p+2q,0]$, $[1,p+2q-2,1]$, $\ldots$, $[q,p,q]$. Then we have to show that the number of such monomials matches the total number of operators we constructed in these representations.

Let us illustrate how this works with an example. Consider the $[2,2,2]$ representation. The complete set of scalar composite operators we can build out of 4 z_1's and 2 z_2's is

$$
\begin{aligned}
6:&\quad \mathrm{tr}\, z_1z_1z_1z_1z_2z_2,\ \ \mathrm{tr}\, z_1z_1z_1z_2z_1z_2,\ \ \mathrm{tr}\, z_1z_1z_2z_1z_1z_2\\
4+2:&\quad \mathrm{tr}\, z_1z_1z_1z_1\,\mathrm{tr}\, z_2z_2,\ \ \mathrm{tr}\, z_1z_1z_1z_2\,\mathrm{tr}\, z_1z_2,\ \ \mathrm{tr}\, z_1z_1z_2z_2\,\mathrm{tr}\, z_1z_1,\ \ \mathrm{tr}\, z_1z_2z_1z_2\,\mathrm{tr}\, z_1z_1\\
3+3:&\quad \mathrm{tr}\, z_1z_1z_1\,\mathrm{tr}\, z_1z_2z_2,\ \ \mathrm{tr}\, z_1z_1z_2\,\mathrm{tr}\, z_1z_1z_2\\
2+2+2:&\quad \mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_2z_2,\ \ \mathrm{tr}\, z_1z_1\,\mathrm{tr}\, z_1z_2\,\mathrm{tr}\, z_1z_2
\end{aligned}
\tag{3.47}
$$

or the total of 11 operators. By taking linear combinations of these, we can construct:

- 4 totally symmetric tensors in the $[0,6,0]$ corresponding to the partitions of 6 in (3.47);
- 1 tensor in the representation $[1,4,1]$, given in (3.6);
- 6 tensors in the representation $[2,2,2]$, listed in (3.23).

Thus there are no other scalar composite operators of the form $[(z_1)^4\,(z_2)^2]$.

In the same fashion, we can go through all other representations we have considered in this paper and verify that we didn't leave out any operators.

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